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The free surface on a liquid filling a trench heated from its side

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In this paper we compute the motion and the shape of the free surface on a liquid in a trench heated from its side. The analysis is based on Joseph's Lagrangian theory of domain perturbations, which is developed in general and through simple examples, chosen so as to make the comparison of the Lagrangian method with Stokes's Eulerian theory very clear. The perturbation problems are resolved analytically by application of biorthogonality conditions to a powerful set of biharmonic eigenfunctions.

1. Introduction

When the side walls of an open trench filled with liquid are maintained at unequal temperatures, the liquid will circulate owing to the driving action of buoyant forces induced by density variations. The motion in the trench distorts the free surface. We are interested in finding mathematical expressions to describe the motion and the shape of the free surface. We are going to construct the solution of this basic problem as power series in the temperature difference across the trench, which is pivoted about the state of rest that prevails when this temperature difference is zero.

There are several variations of our basic problem. In the first, we shall consider an infinitely deep rectangular trench, which is filled with liquid right up to the top. The second is the same as the first, except that the trench has a flat bottom. In these two variations the affinity which liquids have for sharp edges is idealized by requiring that the free surface pass through the edges at the top of the trench. (See figure 6.) Our methods work equally well when the trench is partially filled with water; but then, to make things work out simply; we must prescribe an angle of horizontal contact.

To study the free-surface problem, we use the Lagrangian theory of domain perturbations. This theory can be used to find solutions of boundary-value problems, like the membrane problems studied by Hadamard (1908), in which the perturbed domains are prescribed but the solutions are not; and the method can be used for free-surface problems, like the water-wave problems studied by Stokes or like the problem studied here, in which the shape of the free surface is to be determined. The Lagrangian formulation of theory of domain perturbations was developed by Joseph in a series of papers starting in 1967. In the most recent formulations (1973, 1974), one imagines a one-parameter family of domains, on which the field equations and boundary conditions are to be solved. The solution is presumed to be known in some reference domain. It is convenient, but not necessary, to imagine that the solution in the perturbed domain can be developed in a power series in the parameter, the coefficients being substantial derivatives of the field variables evaluated in the reference domain. To compute these derivatives, the deformed domain is first mapped onto a reference domain; in the end, the mapping is inverted, and the derivatives of the solution with respect to the parameter are computed in the deformed domain.

Joseph's theory is called Lagrangian, because it relies heavily on a mapping which is analogous to a material mapping; but, of course, a domain mapping is not necessarily a material mapping. Stokes's (1847) theory† can, in the same spirit, be called 'Eulerian', since that theory can be interpreted as applying always to the motion in the deformed domain. In fact, it is very hard to say very much about Stokes's theory, because some parts of it are worked in the deformed domain, and some in the reference domain. The domain of definition of the functions is almost never defined, neither is it intuitively obvious. Readers who come to this paper with just a little experience of the literature on water waves will immediately recognize the basic procedure of Stokes's theory. First the solution is expanded in powers of ϵ in the deformed domain. This leads to the governing equation, which in water-wave theory is usually taken as Laplace's equation.‡ The boundary conditions are then expanded in a power series in the space variables around some mean position, say a plane. The boundary conditions are then applied on the mean surface, whether or not there is water there, and without regard for the fact that the derivation of the equations has already implied a different domain of definition for the functions. To our knowledge, these difficulties, which are perhaps better recognized by amateurs than by experts, who after long years of practice may achieve perfect insensitivity to difficulties, were first discussed in Joseph (1973).§ In the 1973 paper, it was shown

† See Stoker (1957) or Wehausen & Laitone (1960) for popular and authorative accounts of Stokes's theory. It is no surprise that Stokes's original paper is not less clear than later expositions of his theory. But Stokes's is not surpassingly clear.

‡ We want to stress the obvious fact that Laplace's equation arises only in a special case. When you have Laplace's equation in two dimensions, you can do a lot of beautiful things with complex variables that cannot be done generally. It is no surprise that existence proofs for wave problems are nearly all confined to the complex variable case $x+iy = f(\phi+i\psi)$, which reduces the problem to a fixed domain. (See Levi-Civita 1925; Struik 1926.) These proofs are beside the point here. They apply only in a too special situation; in any case, they have only the most oblique connexion with Stokes's theory. On the other hand, Sattinger's (1975) proof, of convergence of the *Lagrangian* series for the free surface on a Newtonian liquid between cylinders rotating at different speeds (Joseph & Fosdick 1973), is very much to the point. Sattinger's demonstration shows that the series converge, and the solution is regular at the corner when a flat contact angle is prescribed at the boundary. Questions of convergence and regularity of the solution when the contact is other than flat are still open.

§ A summary of difficulties in interpreting Stokes's theory follows. (i) The derivation of the governing equations leaves the domain of the functions ambiguous. In the end, only functions which are defined in the reference configuration are computed. (ii) There is no prescription about how to compute the solution in the deformed domain where the solution is wanted.

that Stokes's theory was consistent with Joseph's, given a certain analytic continuation of functions originally defined on the reference domain into the deformed domain. In this paper, we are going to show that the substantial derivatives of the solution following the mapping can be computed in the deformed domain in a simple and natural way by merely inverting the mapping, and that analytic continuation of functions defined on the reference configuration is never required. By using the mapping principle, we can calculate several solutions in the physical domain from a single solution in the reference domain. (See figure 6.)

The Lagrangian theory of domain perturbations is developed in the context of the trench problem in § 3. There are two roads from § 3. One goes to the trench problem. The other goes to an extremely simple made-up problem (in the appendix), designed to yield the main ideas of the mapping principle without a lot of complicated equations. Mathematically, the made-up problem is in no way inferior to any other; it is just simpler.

The application of the domain perturbation theory to the trench problem leads us, in § 4, to a biharmonic problem, related to the problem of computing stresses in a thin semi-infinite strip clamped at its sides. The stress problem has been solved in terms of a 'Fourier' series of biorthogonal eigenfunctions (the Papkovich–Fadle functions) by Smith (1952) and by Johnson & Little (1965). The methods used by these authors are very powerful in solving edge problems that arise in fluid-filled cavities with and without free surfaces. Smith's formulation is particularly convenient, and has been extended by Joseph (1974) to the cylindrical equivalent of the edge problem that arises in the study of the free surface on the edge of the liquid filling a torsion flow viscometer. Despite the apparently great differences between our problem and the torsion flow problem, they are mathematically alike, and in special limits they coincide. We believe that our application of these methods to the problem of a free surface on a liquid filling a trench of finite depth is new, and could be applied to finite-strip problems arising in the linear theory of elasticity. We find that the top and bottom of a trench of finite depth do not interact, when the depth-to-width ratio is greater than about 3.

2. Mathematical formulation

We consider the steady motion of a Newtonian liquid in a long open trench of width d and length L. The liquid fills the trench up to a mean height D; i.e. the total volume of the liquid is DdL. We let $L \rightarrow \infty$, and treat a two-dimensional problem in a rectangle, which is open at the top. Co-ordinates (x, z) are located relative to a centre midway between the walls at a height D from a flat bottom. The pressure of the atmosphere at the top of the trench is p_a, and the reduced pressure in the fluid is designated as

$$\Phi = p(x, z) - p_a - \rho g z.$$

$p(x, z)$ is the pressure in the fluid, ρ is the density and g is the acceleration due to gravity. Since the fluid is incompressible, the area Dd occupied by the fluid is an invariant, and the free-surface height

$$z = h(x)$$

must have zero mean value

$$0 = \frac{1}{d}\int_{-\frac{1}{2}d}^{\frac{1}{2}d} h(x)\,dx.$$

The temperatures on the walls at $x = \frac{1}{2}d$ and $x = -\frac{1}{2}d$ are $T_0+\epsilon$ and T_0, respectively. We shall assume that the bottom of the trench is insulated; and, since the thermal conductivity of liquids is generally so much greater than that of air, the free surface of the liquid–air interface is also assumed to be insulated. We designate the temperature as $T(x,z) = \theta(x,z)+T_0$, the solenoidal velocity by $\mathbf{u} = \mathbf{e}_x u + \mathbf{e}_z w$, the stress deviator by $\mathbf{S} = 2\mu\mathbf{D}[\mathbf{u}]$, where $\mathbf{D}$ is the stretching ('rate of strain') tensor and μ is the viscosity. The variables $\theta(x,z)$, $\mathbf{u}(x,z)$ and $\Phi(x,z)$ are defined on

$$\mathscr{V}_\epsilon = \mathscr{V}_\epsilon[x,z|;\ -\tfrac{1}{2}d \leqslant x \leqslant \tfrac{1}{2}d,\ -D \leqslant z \leqslant h(x;\epsilon)].$$

The dependence of the free surface $h(x;\epsilon)$ on the temperature difference ϵ is to be determined. On the free surface, the normal component of the velocity vanishes: $\mathbf{u}.\mathbf{n} = 0$. The normal component of the temperature gradient vanishes: $\mathbf{n}.\nabla T = 0$. The shear stress vanishes: $S_{nt} = 0$. The normal component of the stress jump is balanced by surface tension

$$S_{nn}-p+p_a = S_{nn}-\Phi+\rho gh = \sigma[h'/(1+h'^2)^{\frac{1}{2}}]'.$$

We shall assume that the fluid sticks to a sharp edge or makes a flat contact with the vertical walls: $h(\pm\frac{1}{2}d) = 0$ or $h'(\pm\frac{1}{2}d) = 0$. The Oberbeck–Boussinesq equations† are assumed to govern the motion in $\mathscr{V}_\epsilon$:

$$\rho\mathbf{u}.\nabla\mathbf{u} = -\nabla\Phi+\mathbf{e}_z\rho g\alpha\theta+\mu\nabla^2\mathbf{u}, \quad \mathbf{u}.\nabla\theta = \kappa\nabla^2\theta. \tag{2.1}$$

α and μ are the coefficients of thermal expansion and diffusion, respectively.

Now we shall summarize the governing equations. As a notational convenience, we define two sets of functions A_ϵ and B:

$$A_\epsilon = [\mathbf{u}(x,z),\theta(x,z)|\nabla.\mathbf{u} = 0 \quad \text{in} \quad \mathscr{V}_\epsilon,\ \mathbf{u}(\pm\tfrac{1}{2}d,z) = 0,$$
$$\mathbf{u}(x,-D) = 0, \quad \theta(-\tfrac{1}{2}d,z) = 0, \quad \partial\theta/\partial z|_{z=-D} = 0],$$

$$B = \left[h(x)|\, 0 = \frac{1}{d}\int_{-\frac{1}{2}d}^{\frac{1}{2}d} h(x)\,dx, \quad h(\pm\tfrac{1}{2}d) = 0 \quad \text{or} \quad h'(\pm\tfrac{1}{2}d) = 0\right].$$

We shall seek solutions

$$[\mathbf{u}(x,z),\theta(x,z)] \quad \text{in} \quad A_\epsilon \quad \text{and} \quad \Phi(x,z) \tag{2.2}$$

of (2.1), for which

$$\theta(\tfrac{1}{2}d,z;\epsilon) = \epsilon, \tag{2.3}$$

and, on $z = h(x;\epsilon)$,

$$\frac{\partial\theta}{\partial z}-h'\frac{\partial\theta}{\partial x} = 0, \quad w-h'u = 0, \tag{2.4}$$

$$(S_{zz}-S_{xx})h'+(1-h'^2)S_{xz} = 0, \quad S_{zz}-h'S_{zx}-\Phi = \sigma[h'/(1+h'^2)^{\frac{1}{2}}]'-\rho gh, \tag{2.5}$$

where

$$h\in B. \tag{2.6}$$

† Although a Newtonian fluid is assumed, any rheologically complex fluid, whose stress is expandable in a power series of Rivlin–Ericksen tensors, could just as well have been used. All such fluids are Newtonian to the order to which our calculations have been carried out.

The problem (2.1)–(2.6) is nearly intractable, because it is nonlinear and is posed on a domain $\mathscr{V}_\epsilon$ whose shape must be determined as part of the solution.

3. The Lagrangian theory of domain perturbations

To study the problem (2.1)–(2.6) when $\epsilon \neq 0$, we first map $\mathscr{V}_\epsilon$ onto the reference domain $\mathscr{V}_0$ of the rest solution. We then expand the mapped problem in a power series, determine the coefficients of the series relative to the rest state, and discuss the problem of representing the solution in the deformed domain where the physical problem is defined. The discussion is best carried out in three parts: properties of the mapping (§ 3.1); the series solution, and the unique determination of the boundary values of the mapping (§ 3.2); representations of the solution in $\mathscr{V}_\epsilon$ (§ 3.3).

3.1. *Properties of the mapping*

We define the mapping $(x, z) \rightarrow (x_0, z_0)$ such that

$$x = x_0, \quad -\tfrac{1}{2}d \leqslant x \leqslant \tfrac{1}{2}d; \tag{3.1a}$$

$$z = \mathscr{z}(x_0, z_0, D; \epsilon), \quad -D \leqslant z \leqslant h(x, D; \epsilon), \quad -D \leqslant z_0 \leqslant 0. \tag{3.1b}$$

The mapping (3.1) is to be one-to-one, carrying boundary points to boundary points:

$$h(x, D; \epsilon) = \mathscr{z}(x_0, 0, D; \epsilon) \tag{3.1c}$$

and

$$-D = \mathscr{z}(x_0, -D; D; \epsilon). \tag{3.1d}$$

We also want the mapping to be analytic in ϵ and uniquely invertible:

$$z_0 = \mathscr{z}_0(x, z, D; \epsilon). \tag{3.1e}$$

This is a very general mapping of $\mathscr{V}_\epsilon \leftrightarrow \mathscr{V}_0$, and it is not uniquely determined; we could work the theory for yet more general mappings. The strong requirement of analyticity can be relaxed for approximate solutions; then we would consider asymptotic representations with truncated power series.

We are going to show (in § 3.2) that the perturbation problems determine a unique function $h(x, D; \epsilon)$, and that the solution in the deformed domain is independent of the choice of the function $\mathscr{z}$. Given the uniqueness of h, we are assured that it is possible to construct and use the linear *scaling transformation*

$$\mathscr{z}(x_0, z_0, D; \epsilon) = h(x_0, D; \epsilon) + [h(x_0, D; \epsilon) + D]\, z_0/D. \tag{3.2a}$$

This transformation has a simple form when the co-ordinates are reckoned from the bottom of the trench. If $\mathscr{z} = y - D$, $z_0 = y_0 - D$, $\mathscr{h} = h - D$, then

$$y = y_0 \mathscr{h}/D. \tag{3.2b}$$

This apparently special mapping is homogeneous of degree one in the variables h, z_0 and D; depends on x_0 and ϵ through h; and has the property that

$$z_0 = \frac{1}{d}\int_{-\frac{1}{2}d}^{\frac{1}{2}d} \mathscr{z}(x_0, z_0, D; \epsilon)\, dx_0$$

is the mean value of z. In the limit $D \to \infty$ the linear *scaling transformation* reduces to

$$z(x_0, z_0; \epsilon) = h(x_0; \epsilon) + z_0. \tag{3.3}$$

We call (3.3) a linear *shifting transformation.*

3.2. *The series solution and unique determination of the boundary values of the mapping*

Functions $\psi(x, z; \epsilon)$ defined in $\mathscr{V}_\epsilon$ are of the form $\psi(x_0, z(x_0, z_0, D; \epsilon); \epsilon)$ when mapped onto $\mathscr{V}_0$ by (3.1). For these mapped functions, we define the partial derivative holding $z(x_0, z_0, D; \epsilon)$ fixed:

$$\psi^{\langle n \rangle}(x_0, z_0) = \left.\frac{\partial^n \psi}{\partial \epsilon^n}\right|_{\epsilon=0}.$$

We also define a substantial derivative following the mapping:

$$\left.\begin{aligned}
\psi^{[n]}(x_0, z) &= \left.\frac{d^n \psi}{d\epsilon^n}\right|_{\epsilon=0},\\
\psi^{[0]} &= \psi^{\langle 0 \rangle},\\
\psi^{[1]} &= \psi^{\langle 1 \rangle} + z^{[1]}\psi^{\langle 0 \rangle}_{,z},\\
\psi^{[2]} &= \psi^{\langle 2 \rangle} + 2z^{[1]}\psi^{\langle 1 \rangle}_{,z} + z^{[2]}\psi^{\langle 0 \rangle}_{,z} + z^{[1]^2}\psi^{\langle 0 \rangle}_{,zz},\\
\psi^{[3]} &= \psi^{\langle 3 \rangle} + 3z^{[2]}\psi^{\langle 1 \rangle}_{,z} + 3z^{[1]}\psi^{\langle 2 \rangle}_{,z} + z^{[3]}\psi^{\langle 0 \rangle}_{,z}\\
&\quad + 3z^{[1]^2}\psi^{\langle 1 \rangle}_{,zz} + 3z^{[1]}z^{[2]}\psi^{\langle 0 \rangle}_{,zz} + z^{[1]^3}\psi^{\langle 0 \rangle}_{,zzz},
\end{aligned}\right\} \tag{3.4}$$

etc. Since x_0 and z_0 do not depend on ϵ, and $z = z(x_0, z_0; D; \epsilon)$, we have

$$z^{[n]} = z^{\langle n \rangle} \quad \text{and} \quad h^{[n]} = h^{\langle n \rangle}.$$

The substantial derivative is an important operator in the Lagrangian theory, because solutions defined in the mapped domain are to be expressed in a power series, whose coefficients are substantial derivatives evaluated in $\mathscr{V}_0$:

$$\begin{bmatrix} \mathbf{u}(x, z; \epsilon) \\ \theta(x, z; \epsilon) \\ \Phi(x, z; \epsilon) \\ h(x; \epsilon) \end{bmatrix} = \sum_{n=0} \frac{1}{n!} \begin{bmatrix} \mathbf{u}^{[n]}(x_0, z_0) \\ \theta^{[n]}(x_0, z_0) \\ \Phi^{[n]}(x_0, z_0) \\ h^{[n]}(x_0) \end{bmatrix} \epsilon^n. \tag{3.5}$$

Dependence on the parameter D has been suppressed. The boundary-value problems satisfied by coefficients in (3.5) are very complicated (see Joseph 1967); but a major simplification of them is possible (Joseph 1973). This simplification allows one to compute the partial derivatives $[\mathbf{u}^{\langle n \rangle}, \theta^{\langle n \rangle}, \Phi^{\langle n \rangle}, h^{\langle n \rangle}]$ from much simpler problems. We discuss this simplification next.

We first note that the field equations (2.1) and $\nabla . \mathbf{u} = 0$ hold in $\mathscr{V}_\epsilon$ and for an ϵ interval around $\epsilon = 0$. For example, consider the equation $\nabla . \mathbf{u} = 0$. Since this is an identity in ϵ, for small ϵ

$$\frac{d^n}{d\epsilon^n}(\nabla . \mathbf{u}) = 0.$$

Since it is also an identity in $\mathscr{V}_\epsilon$, we have that, for each and every integer $n \geqslant 0$,

$$\frac{d^n}{d\epsilon^n}(\nabla . \mathbf{u}) = \nabla . (\partial^n \mathbf{u}/\partial \epsilon^n) = 0. \tag{3.6}$$

To prove (3.6), we shall show that, if it holds for some integer $l = n$, then it also holds when $l = n+1$:

$$\frac{d^{n+1}}{d\epsilon^{n+1}}(\nabla.\mathbf{u}) = \frac{d}{d\epsilon}\nabla.(\partial^n\mathbf{u}/\partial\epsilon^n)$$

$$= \frac{\partial}{\partial\epsilon}\nabla.(\partial^n\mathbf{u}/\partial\epsilon^n) + \frac{dz}{d\epsilon}\frac{\partial}{\partial z}\nabla.(\partial^n u/\partial\epsilon^n)$$

$$= \nabla.(\partial^{n+1}\mathbf{u}/\partial\epsilon^{n+1}) = 0. \tag{3.7}$$

Since (3.7) holds when $n = 0$, induction proves that (3.6) holds for all n. The same kind of simplification applies to any equation simultaneously an identity in ϵ and in $z(-D \leqslant z \leqslant h(x;\epsilon))$. This observation, and the remark that $h^{[n]} = h^{\langle n\rangle}$ is enough to establish that

$$\theta^{\langle n\rangle}(d/2, z_0) = \delta_{n1}, \tag{3.8a}$$

and almost enough to establish that

$$[\mathbf{u}^{\langle n\rangle}(x_0, z_0), \theta^{\langle n\rangle}(x_0, z_0)] \in A_0. \tag{3.8b}$$

To establish (3.8*b*) we must show that

$$\partial\theta^{\langle n\rangle}(x_0, z_0)/\partial z_0 = 0 \quad \text{on} \quad z = -D.$$

This follows from induction using the fact that

$$\frac{d}{d\epsilon}(.) = (.)^{\langle 1\rangle} + z^{[1]}(.)^{\langle 0\rangle} = (.)^{\langle 1\rangle},$$

because $z = -D$ when $z_0 = -D$ and $z^{[n]} = 0$ when $z_0 = -D$. The differential equations that follow from (2.1) using the induction (3.7) are

$$\rho(\mathbf{u}.\nabla\mathbf{u})^{\langle n\rangle} = -\nabla\Phi^{\langle n\rangle} + \mathbf{e}_z\,\rho\alpha g\theta^{\langle n\rangle} + \mu\nabla^2\mathbf{u}^{\langle n\rangle}, \quad (\mathbf{u}.\nabla\theta)^{\langle n\rangle} = \kappa\nabla^2\theta^{\langle n\rangle} \quad \text{in} \quad \mathscr{V}_0. \tag{3.8c}$$

No such simplification is possible relative to (2.4) and (2.5) on the free surface. These equations are of the form $F(x, h(x,\epsilon)) = 0$ and, of course $\partial F(x,z)/\partial z|_{z=h(x)}$ is not necessarily zero. It follows that free-surface perturbation equations must be written in the form

$$F^{[n]} = \left(\frac{\partial}{\partial\epsilon} + h^{[1]}\frac{\partial}{\partial z_0}\right)^n F|_{\epsilon=0} = 0. \tag{3.8d}$$

Repeated application of $\partial/\partial\epsilon\, h^{[n]} = h^{[n+1]}$ in (3.8*d*) shows

$$F^{(n)} = F^{\langle n\rangle} + \ldots h^{[n]}F^{\langle 0\rangle},$$

where

$$h^{[n]} = h^{\langle n\rangle} \in B. \tag{3.8e}$$

Equations (3.8) are boundary-value problems for the partial derivatives $[\mathbf{u}^{\langle n\rangle}, \theta^{\langle n\rangle}, \Phi^{\langle n\rangle}, h^{\langle n\rangle}]$ in the flat reference domain. It is easy to show that *these functions can be determined sequentially.*

When $n = 0$ we have the null solution

$$[\mathbf{u}^{[0]}, \theta^{[0]}, \Phi^{[0]}, h^{[0]}] = [u^{\langle 0\rangle}, \theta^{\langle 0\rangle}, \Phi^{\langle 0\rangle}, h^{\langle 0\rangle}] = 0 \tag{3.9}$$

of the rest state. Therefore, the series (3.5) can be started with $n = 1$. When $n = 1$ we have

$$[\mathbf{u}^{\langle 1\rangle}(x_0, z_0), \theta^{\langle 1\rangle}(x_0, z_0)] \in A_0, \tag{3.10}$$

$$0 = -\nabla\Phi^{\langle 1\rangle} + \mathbf{e}_z \rho\alpha g\theta^{\langle 1\rangle} + \mu\nabla^2\mathbf{u}^{\langle 1\rangle}, \quad 0 = \kappa\nabla^2\theta^{\langle 1\rangle} \quad \text{in} \quad \mathscr{V}_0, \tag{3.11a,b}$$

$$\theta^{\langle 1\rangle}(\tfrac{1}{2}d, z_0) = 1, \quad -D \leqslant z_0 \leqslant 0. \tag{3.12}$$

On the free surface, the equations simplify, because of the vanishing of the solution at zeroth order. For example, the normal component of velocity vanishes on the free surface at first order if

$$w^{[1]} = w^{\langle 1\rangle} - h^{\langle 1\rangle}u^{\langle 0\rangle} = w^{\langle 1\rangle} = 0. \tag{3.13a}$$

A similar computation for the shear stress gives

$$S_{zx}^{\langle 1\rangle} = \mu\left[\frac{\partial u^{\langle 1\rangle}}{\partial z_0} + \frac{\partial w^{\langle 1\rangle}}{\partial x_0}\right] = 0 \tag{3.13b}$$

on $z_0 = 0$.

The boundary-value problem (3.10)–(3.13) is uniquely solvable, and does not depend on any of the derivatives of h. We are going to solve this problem in § 5. Once we have obtained the solution, we may find $h^{\langle 1\rangle}$ by integrating the normal stress jump condition

$$2\mu\frac{\partial w^{\langle 1\rangle}}{\partial z_0} - \Phi^{\langle 1\rangle} = \sigma\frac{d^2h^{\langle 1\rangle}}{dx_0^2} - \rho g h^{\langle 1\rangle}, \tag{3.14a}$$

where

$$h^{\langle 1\rangle} \in B. \tag{3.14b}$$

3.3. *Representations of the solution in* $\mathscr{V}_\epsilon$

The equations for the partial derivatives $(\mathbf{u}^{\langle n\rangle}, \theta^{\langle n\rangle}, \Phi^{\langle n\rangle}, h^{\langle n\rangle})$ can be obtained by the method of Stokes, which was described in § 1. We are not aware of an explicit discussion, other than that of Joseph (1973), of how to represent the solution in $\mathscr{V}_\epsilon$, given the partial derivatives defined in $\mathscr{V}_0$. The nature of Stokes's derivations is such that it is possible that he imagined the solutions in $\mathscr{V}_\epsilon$ to be given by

$$\begin{bmatrix}\mathbf{u}(x,z;\,\epsilon)\\ \theta(x,z;\,\epsilon)\\ \Phi(x,z;\,\epsilon)\\ h(x;\,\epsilon)\end{bmatrix} = \sum_{n=1}\frac{1}{n!}\,\epsilon^n\begin{bmatrix}\mathbf{u}^{\langle n\rangle}(x,z)\\ \theta^{\langle n\rangle}(x,z)\\ \Phi^{\langle n\rangle}(x,z)\\ h^{\langle n\rangle}(x)\end{bmatrix}. \tag{3.15}$$

The functions $\mathbf{u}^{\langle n\rangle}(.,.)$, $\theta^{\langle n\rangle}(.,.)$ and $\Phi^{\langle n\rangle}(.,.)$ are the functions first found in $\mathscr{V}_0$, then continued analytically, by declaring that $(x_0, z_0) \to (x, z)$, where $-D \leqslant z \leqslant h(x,\epsilon)$.

In fact, the representation (3.15) is implied by (3.5). (See Joseph 1973.) To show this, it is necessary to replace the substantial derivatives in (3.5) with the expressions (3.4), then to rearrange the series, as follows:

$$\begin{aligned}
&\Phi^{[0]}(x_0, z_0;\,0) + \epsilon\Phi^{[1]} + \tfrac{1}{2}\epsilon^2\Phi^{[2]} + \epsilon^3\Phi^{[3]}/3! + \ldots\\
&\quad = \Phi^{\langle 0\rangle}(x_0, z_0;\,0) + \epsilon z^{[1]}\Phi^{\langle 0\rangle}_{,z} + \epsilon^2(z^{[2]}\Phi^{\langle 0\rangle}_{,z} + z^{[1]^2}\Phi^{\langle 0\rangle}_{,zz})/2!\\
&\qquad + \epsilon^3(z^{[3]}\Phi^{\langle 0\rangle}_{,z} + 3z^{[1]}z^{[2]}\Phi^{\langle 0\rangle}_{,zz} + z^{[1]^3}\Phi^{\langle 0\rangle}_{,zzz})/3! + \ldots + \epsilon\{\Phi^{\langle 1\rangle}(x_0, z_0;\,0)\\
&\qquad + \epsilon z^{[1]}\Phi^{\langle 1\rangle}_{,z} + \epsilon^2(z^{[2]}\Phi^{\langle 1\rangle}_{,z} + z^{[1]^2}\Phi^{\langle 1\rangle}_{,zz}/2!) + \ldots\}\\
&\qquad + \tfrac{1}{2}\epsilon^2\{\Phi^{\langle 2\rangle}(x_0, z_0;\,0) + \epsilon z^{[1]}\Phi^{\langle 2\rangle}_{,z} + \ldots\} + \ldots\\
&\quad = \Phi^{\langle 0\rangle}(x, z(x_0, z_0;\,\epsilon);\,0) + \epsilon\Phi^{\langle 1\rangle}(x, z(x_0, z_0;\,\epsilon);\,0) + \tfrac{1}{2}\epsilon^2\Phi^{\langle 2\rangle}(x, z(x_0, z_0;\,\epsilon);\,0) + \ldots\\
&\quad = \Phi(x, z;\,\epsilon).
\end{aligned}$$

It follows that, given the partial derivatives in the domain $\mathscr{V}_0$, we may compute the solution in $\mathscr{V}_\epsilon$ by the analytic continuation of the functions leading to (3.15), or by direct inversion of the mapping function in the arguments of the substantial derivative in (3.5). For example, using the linear *scaling transformation* (3.2), we find that

$$\begin{bmatrix} \mathbf{u}(x,z;\,\epsilon) \\ \theta(x,z;\,\epsilon) \\ \Phi(x,z;\,\epsilon) \\ h(x;\,\epsilon) \end{bmatrix} = \sum_{n=1} \frac{\epsilon^n}{n!} \begin{bmatrix} \mathbf{u}^{[n]}\{x,(z-h)\,D/(D+h)\} \\ \theta^{[n]}\{x,(z-h)\,D/(D+h)\} \\ \Phi^{[n]}\{x,(z-h)\,D/(D+h)\} \\ h^{\langle n\rangle}(x) \end{bmatrix}. \tag{3.16}$$

Assuming convergence, the series on right of (3.15) and (3.16) sum to the same function, but partial sums after N terms are different. This difference leaves open the question of the optimal representation for approximate solutions; and we have not answered this question. But we have found that the representation (3.16) is particularly convenient, for the following reasons. (i) The series (3.16) retains the same basic ordering in powers of ϵ as the series (3.5), which gives the solution in a fully expanded form. (ii) The inversion of the mapping does not require that the functions $\mathbf{u}^{[n]}$, $\theta^{[n]}$ or $\Phi^{[n]}$ should be continued. For example, the domain of the function

$$\mathbf{u}^{[n]}\{x,(z-h)\,D/(D+h)\} = \mathbf{u}^{[n]}(x_0,z_0) \tag{3.17}$$

is unchanged under the inversion of the mapping

$$-D \leqslant (z-h)\,D/D+h \leqslant 0.$$

(iii) The level lines of the coefficients of (3.16) in $\mathscr{V}_\epsilon$ bear a simple scaling relation to the level lines calculated in $\mathscr{V}_0$. In particular, under the inversion of the mapping the level lines of, say, $\mathbf{u}^{[n]}(x_0,z_0)$ continue to conform to the boundary. (See figure 6.) In contrast, the level lines of the continued function

$$\mathbf{u}^{\langle n\rangle}(x,z), \quad -D \leqslant z \leqslant h(x;\,\epsilon),$$

bear no simple relation to the level lines of the computed functions

$$\mathbf{u}^{\langle n\rangle}(x_0,z_0), \quad -D \leqslant z_0 \leqslant 0.$$

The analysis of this section, and the comparison of the representations (3.15) and (3.16) may be more easily understood by studying the simple example given in the appendix.

4. Computation of the motion in the trench

We return now to the problem (3.10)–(3.13), governing the slow motion of the fluid in the trench. The equations for the temperature do not depend on the velocity, and they may be solved separately:

$$\theta^{\langle 1\rangle} = \tfrac{1}{2} + x_0/d. \tag{4.1}$$

We next calculate the velocity field by substituting (4.1) back into (3.11 *a*), and converting the remaining part of the problem into a biharmonic edge problem of the type solved by Smith (1952). Introducing a stream function ψ,

$$u^{\langle 1\rangle} = \psi_{,z_0}, \quad w^{\langle 1\rangle} = -\psi_{,x_0},$$

into (3.11 a), we find that

$$\Phi_{,x_0} = \mu\nabla^2\psi_{,z_0}, \quad \Phi_{,z_0} = \rho g\alpha(x_0/d+\tfrac{1}{2}) - \mu\nabla^2\psi_{,x_0}.$$

Elimination of Φ between these two equations leads to

$$\nabla^4\psi = \rho g\alpha/(\mu d) \quad \text{in} \quad \mathscr{V}_0, \tag{4.2a}$$

where
$$\psi(\pm\tfrac{1}{2}d, z_0) = \psi_{,x_0}(\pm\tfrac{1}{2}d, z_0) = \psi(x_0, -D) = \psi_{,z_0}(x_0, -D) = \psi(x_0, 0) = \psi_{,z_0z_0}(x_0, 0) = 0. \tag{4.2b}$$

To reduce this problem to an edge problem, we introduce new (dimensionless) variables

$$(t,y) = \frac{2}{d}(x_0, z_0), \quad \tilde{D} = \frac{2}{d}D, \quad \Psi = \frac{\mu}{\rho g\alpha d^3}\psi - \frac{(t^2-1)^2}{384}, \quad \mathscr{P} = \frac{\Phi}{8\rho g\alpha d}.$$

The new variables satisfy the equations

$$\nabla^4\Psi = 0 \quad \text{in} \quad [-1 \leqslant t \leqslant 1, -\tilde{D} \leqslant y \leqslant 0], \tag{4.3a}$$

$$\Psi(\pm 1, y) = \Psi_{,t}(\pm 1, y) = 0, \tag{4.3b}$$

$$\Psi_{,y}(t, -\tilde{D}) = \Psi_{,yy}(t, 0) = 0 \tag{4.3c}$$

and
$$\Psi(t, 0) = \Psi(t, -\tilde{D}) = -(t^2-1)^2/384. \tag{4.3d}$$

Given Ψ, the dimensionless pressure $\mathscr{P}$ may be found from

$$\mathscr{P}_{,t} = \tfrac{1}{2}(\Psi_{,ytt} + \Psi_{,yyy}), \quad \mathscr{P}_{,y} = \tfrac{1}{32} - \tfrac{1}{2}(\Psi_{,ttt} + \Psi_{,tyy}). \tag{4.4}$$

The solution of the biharmonic problem (4.3) is given by

$$\Psi = \lim_{N\to\infty} \sum_{-N}^{N} [C_n \exp(S_n y) + D_n \exp(-S_n y)]\, \phi_1^{(n)}(t)/S_n^2, \tag{4.5a}$$

where $C_0 = D_0 = 0$. The S_n ($n = 1, 2, \ldots$) are the first-quadrant complex roots of

$$\sin 2S = -2S \tag{4.5b}$$

numbered in a sequence corresponding to increasing size of the real part of S_n and $S_{-n} = \bar{S}_n$. The functions $\phi_1^{(n)}(t)$ are biharmonic eigenfunctions (Papkovich–Fadle functions), belonging to the eigenvalues S_n. These functions are given by (4.12); they satisfy the side-wall boundary conditions (4.3 b) when the S_n are the roots of (4.5 b). The coefficients C_n and D_n are to be chosen so as to make (4.5 a) satisfy the edge conditions (4.3 c, d). The roots of $\sin 2S = -2S$ are symmetrically located in the complex S plane, and the above numbering covers all roots. It follows from (4.7) and (4.12) that

$$\phi_1^{(n)}(t) = \bar{\phi}_1^{(-n)}(t) \quad (n = \pm 1, \pm 2, \ldots). \tag{4.5c}$$

Since the given edge data are real,

$$C_n = \bar{C}_{-n}, \quad D_n = \bar{D}_{-n} \quad (n = \pm 1, \pm 2, \ldots). \tag{4.5d}$$

The problem defined by (4.3) when the trench is infinitely deep ($\tilde{D} \to \infty$) is identical to that solved by Smith (1952). It is instructive to give the solution for the infinitely deep trench first, then to apply Smith's method to find the solution for the trench of finite depth.

4.1. *The infinitely deep trench*

When $\tilde{D} \to \infty$, we replace the boundary conditions on $y = -\tilde{D}$ with the requirement that ψ and all its derivatives tend to zero as $y \to -\infty$. To satisfy this condition, we must have $D_n = 0$ ($n = \pm 1, \pm 2, \ldots$). Substituting (4.5*a*) into (4.3 *c, d*) we get

$$\begin{bmatrix} \Psi_{,yy}(t,0) \\ \Psi(t,0) \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{(t^2-1)^2}{384} \end{bmatrix} = \sum_{-\infty}^{\infty} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_1^{(n)}(t)/S_n^2 \end{bmatrix}. \tag{4.6}$$

To put this edge condition into Smith's form, we differentiate the bottom element in the column vector twice, and find that

$$\begin{bmatrix} \Psi_{,yy}(t,0) \\ \Psi_{,tt}(t,0) \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{3t^2-1}{96} \end{bmatrix} = \sum_{-\infty}^{\infty} C_n \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix}, \tag{4.7}$$

where
$$\phi_2^{(n)}(t) = \frac{1}{S_n^2}\frac{d^2}{dt^2}\phi_1^{(n)}.$$

To determine the constants C_n, we introduce the vectors

$$\mathbf{f} = \begin{bmatrix} 0 \\ -\dfrac{3t^2-1}{96} \end{bmatrix}, \quad \boldsymbol{\phi}^{(n)} = \begin{bmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \end{bmatrix}, \quad \boldsymbol{\psi}^{(n)} = \begin{bmatrix} \psi_1^{(n)} \\ \psi_2^{(n)} \end{bmatrix}.$$

$\boldsymbol{\phi}^{(n)}$ and $\boldsymbol{\psi}^{(n)}$ are defined through the differential equations

$$\frac{d^2}{dt^2}\boldsymbol{\phi}^{(n)} + S^2\mathbf{A}\boldsymbol{\phi}^{(n)} = 0, \tag{4.8a}$$

$$\frac{d^2}{dt}\boldsymbol{\psi}^{(n)} + S^2\mathbf{A}^T\boldsymbol{\psi}^{(n)} = 0, \tag{4.8b}$$

where
$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{A}^T = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

The boundary conditions are

$$\phi_1^{(n)} = \frac{d\phi_1^{(n)}}{dt} = \psi_2^{(n)} = \frac{d}{dt}\psi_2^{(n)} = 0 \quad \text{at} \quad t = \pm 1. \tag{4.9}$$

These boundary conditions can be satisfied when the parameter S satisfies (4.5*b*) or (4.13*a*); (4.5*b*) leads to the even eigenfunctions (4.12), whereas (4.13*a*) leads to the odd eigenfunctions (4.13*b*). Using (4.8) and (4.9), we find that

$$\int_{-1}^{1} \boldsymbol{\psi}^{(m)}\mathbf{A}\boldsymbol{\phi}^{(n)}\,dt = k_n\delta_{nm}. \tag{4.10}$$

Applying the biorthogonality condition (4.10) to (4.7), and using (4.8*b*), we find that

$$C_n = \frac{1}{k_n}\int_{-1}^{1} [\psi_1^{(n)}, \psi_2^{(n)}] \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -\dfrac{3t^2-1}{96} \end{bmatrix} dt$$
$$= \frac{1}{k_n}\left(-\frac{1}{96}\right)\int_{-1}^{1} (2\psi_2^{(n)} - \psi_1^{(n)})(3t^2-1)\,dt$$

$$= \frac{1}{96k_n S_n^2}\int_{-1}^{1}\frac{d^2\psi_2^{(n)}}{dt^2}(3t^2-1)\,dt$$

$$= \frac{1}{96k_n S_n^2}\left\{\left[(3t^2-1)\frac{d\psi_2^{(n)}}{dt}\right]_{-1}^{1}-\int_{-1}^{1}6t\frac{d\psi_2^{(n)}}{dt}\,dt\right\}$$

$$= \frac{1}{96k_n S_n^2}\left\{-[6t\psi_2^{(n)}]_{-1}^{1}+\int_{-1}^{1}6\psi_2^{(n)}\,dt\right\}$$

$$= \frac{1}{16k_n S_n^2}\int_{-1}^{1}\psi_2^{(n)}\,dt, \tag{4.11}$$

where

$$\left.\begin{aligned}\phi_1^{(n)} &= S_n\sin S_n\cos S_n t - S_n t\cos S_n\sin S_n t = \psi_2^{(n)},\\ \phi_2^{(n)} &= -(S_n\sin S_n + 2\cos S_n)\cos S_n t + S_n t\cos S_n\sin S_n t,\\ \psi_1^{(n)} &= (S_n\sin S_n - 2\cos S_n)\cos S_n t - S_n t\cos S_n\sin S_n t\end{aligned}\right\} \tag{4.12}$$

are the even eigenfunctions, and $k_n = -4\cos^4 S_n$. When the given data are not even, one must also consider the odd eigenfunctions. For these, the eigenvalues are roots of

$$\sin 2P = 2P, \tag{4.13a}$$

and the eigenfunctions are

$$\left.\begin{aligned}\hat\phi_1^{(n)} &= P_n\cos P_n\sin P_n t - P_n t\sin P_n\cos P_n t = \hat\psi_2^{(n)},\\ \hat\phi_2^{(n)} &= -(P_n\cos P_n - 2\sin P_n)\sin P_n t + P_n t\sin P_n\cos P_n t,\\ \hat\psi_1^{(n)} &= (P_n\cos P_n + 2\sin P_n)\sin P_n t - P_n t\sin P_n\cos P_n t\end{aligned}\right\} \tag{4.13b}$$

and $\hat k_n = -4\sin^4 P_n$.

Finally, after integrating (4.11), we find that

$$C_n = -\tfrac{1}{16}S_n^2\cos^4 S_n. \tag{4.14}$$

The numerical convergence of the partial sum

$$\begin{bmatrix}0\\ \dfrac{(3t^2-1)}{96}\end{bmatrix} = \sum_{-N}^{N}\frac{1}{16S_n^4\cos^4 S_n}\begin{bmatrix}\phi_1^{(n)}(t)\\ \phi_2^{(n)}(t)\end{bmatrix} \tag{4.15}$$

is very rapid (see table 1); and, for practical purposes, the series has converged after three terms. Convergence in the interior is even more rapid. Mathematical convergence of the series (4.15) is easily verified (see Joseph 1974).

In figure 1 we have plotted the level lines of the stream function $\Psi(t,y)$ for the edge problem. These are given by (4.5). We have also shown the streamlines of the flow

$$\frac{\mu}{\rho g\alpha d^3}\psi = \frac{(t^2-1)^2}{384} - \sum_{-\infty}^{\infty}\phi_1^{(n)}(t)\exp(S_n y)/16S_n^4\cos^4 S_n. \tag{4.16}$$

The streamlines are given in the reference domain; they cannot be plotted in the physical domain without first finding the first-order height correction (§ 5). The streamlines in figure 1 show that the edge eddies exist, to turn the stream around at the free edge.

t	$g(t)$	$N = 1$	$N = 3$	$N = 5$	$N = 7$	$N = 9$
1·0	2·083	1·948	2·070	2·079	2·081	2·083
0·8	0·958	0·999	0·950	0·955	0·961	0·960
0·6	0·083	0·110	0·863	0·836	0·818	0·848
0·4	−0·541	−0·548	−0·588	−0·540	−0·543	−0·541
0·2	−0·916	−0·933	−0·922	−0·919	−0·916	−0·916
0	−1·041	−1·057	−1·037	−1·039	−1·040	−1·041

t	$f(t)$	$N = 1$	$N = 3$	$N = 5$	$N = 7$	$N = 9$
1·0	0	0	0	0	0	0
0·8	0	2·986	1·434	0·050	−0·323	−0·128
0·6	0	5·105	−0·8323	0·1061	0·0864	−0·1202
0·4	0	2·916	−0·0656	−0·1830	0·1582	−0·0944
0·2	0	−1·086	+0·5208	0·2174	−0·0278	−0·0718
0	0	−3·003	−0·6155	−0·2265	−0·1105	−0·0631

TABLE 1. Convergence of the Papkovich–Fadle series

$$\begin{bmatrix} f \\ g \end{bmatrix} = 1000 \begin{bmatrix} 0 \\ \dfrac{3t^2-1}{96} \end{bmatrix} \sim \sum_{\substack{-N \\ l \neq 0}}^{N} \frac{1000}{16 S_l^2 \cos^4 S_l} \begin{bmatrix} \phi_1^{(l)}(t) \\ \phi_2^{(l)}(t) \end{bmatrix}$$

The pressure corresponding to (4.5) can be obtained from (4.4) as

$$\mathscr{P} = \frac{y}{32} - \sum_{-\infty}^{\infty} C_n \cos S_n \sin S_n t \exp(S_n y) + A_1. \tag{4.17}$$

A_1 is a constant to be determined from the computation of the free surface.

4.2. *The trench of finite depth*

We are now considering problem (4.3) with $\tilde{D} < \infty$. There is an edge at both the bottom and the top of the trench. There are now two sets of coefficients (C_n and D_n) to be determined from the edge conditions (4.3c, d). At the top ($y = 0$), we find, using (4.5a), that

$$\begin{bmatrix} \psi_{,yy}(t,0) \\ \psi_{,tt}(t,0) \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{3t^2-1}{96} \end{bmatrix} = \sum_{-\infty}^{\infty} \begin{bmatrix} \phi_1^{(n)}(t) \\ \phi_2^{(n)}(t) \end{bmatrix} (C_n + D_n). \tag{4.18}$$

At the bottom ($y = -\tilde{D}$), we have

$$\begin{bmatrix} \psi_{,y}(t,-\tilde{D}) \\ \psi(t,-\tilde{D}) \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{(t^2-1)^2}{384} \end{bmatrix} = \sum_{-\infty}^{\infty} \begin{bmatrix} (C_n \exp(-S_n\tilde{D}) - D_n \exp(S_n\tilde{D}))\,\phi_1^{(n)}(t)/S_n \\ (C_n \exp(-S_n\tilde{D}) + D_n \exp(S_n\tilde{D}))\,\phi_1^{(n)}(t)/S_n^2 \end{bmatrix}.$$

After differentiating the bottom row twice, this can be written as

$$\begin{bmatrix} \psi_{,y}(t,-\tilde{D}) \\ \psi_{,tt}(t,-\tilde{D}) \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{3t^2-1}{96} \end{bmatrix} = \sum_{-\infty}^{\infty} \begin{bmatrix} (C_n \exp(-S_n\tilde{D}) - D_n \exp(S_n\tilde{D}))\,\phi_1^{(n)}(t)/S_n \\ (C_n \exp(-S_n\tilde{D}) + D_n \exp(S_n\tilde{D}))\,\phi_2^{(n)}(t) \end{bmatrix}. \tag{4.19}$$

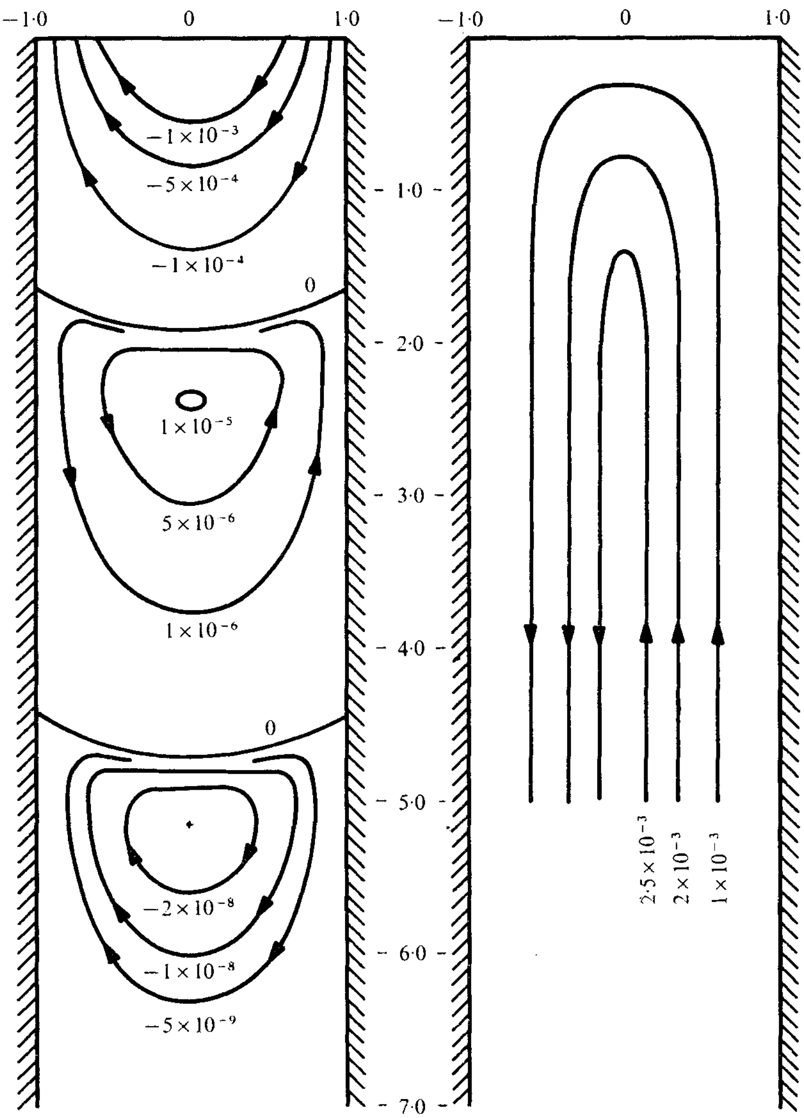

FIGURE 1. Level lines of the edge eddies (4.5) and streamlines (4.16) of the flow in the reference domain for the infinitely deep trench.

The biorthogonality computation, which led to (4.14), now yields

$$C_n + D_n = -\tfrac{1}{16} S_n^2 \cos^4 S_n. \tag{4.20}$$

The same computation, applied to (4.19), where the boundary conditions are of a slightly different type, does not give the coefficient

$$C_n \exp(-S_n \tilde{D}) + D_n \exp(S_n \tilde{D})$$

directly, as in (4.20); instead, it defines that coefficient implicitly through an infinite set of algebraic equations linear in the variables C_n and D_n:

$$\begin{aligned}\frac{1}{4S_n^2} &= \int_{-1}^{1}[\psi_1^{(n)},\psi_2^{(n)}]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\sum_{l=-\infty}^{\infty}\left\{C_l\exp(-S_l\tilde{D})\begin{bmatrix}\phi_1^{(l)}/S_l\\ \phi_2^{(l)}\end{bmatrix}+D_l\exp(S_l\tilde{D})\begin{bmatrix}-\phi_1^{(l)}/S_l\\ \phi_2^{(l)}\end{bmatrix}\right\}\\
&= \sum_{l=-\infty}^{\infty}C_l\exp(-S_l\tilde{D})\int_{-1}^{1}\left\{[\psi_1^{(n)},\psi_2^{(n)}]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\phi_1^{(l)}\\ \phi_2^{(l)}\end{bmatrix}+\frac{1-S_l}{S_l}\psi_2^{(n)}\phi_1^{(l)}\right\}dt\\
&\quad+\sum_{l=-\infty}^{\infty}D_l\exp(S_l\tilde{D})\int_{-1}^{1}\left\{[\psi_1^{(n)},\psi_2^{(n)}]\begin{bmatrix}0 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\phi_1^{(l)}\\ \phi_2^{(l)}\end{bmatrix}-\frac{1+S_l}{S_l}\psi_2^{(n)}\phi_1^{(l)}\right\}dt\\
&= (C_n\exp(-S_n\tilde{D})+D_n\exp(S_n\tilde{D}))(-4\cos^4 S_n)\\
&\quad+\sum_{l=-\infty}^{\infty}\left(C_l\exp(-S_l\tilde{D})-\frac{1+S_l}{1-S_l}D_l\exp(S_l\tilde{D})\right)\frac{1-S_l}{S_l}\int_{-1}^{1}\psi_2^{(n)}\phi_1^{(l)}\,dt\\
&= \sum_{l=-\infty}^{\infty}A_{ln}\left(C_l\exp(-S_l\tilde{D})-\frac{1+S_l}{1-S_l}D_l\exp(S_l\tilde{D})\right)\\
&\quad-4\cos^4 S_n(C_n\exp(-S_n\tilde{D})+D_n\exp(S_n\tilde{D})),\end{aligned}\tag{4.21}$$

where
$$A_{ln}=\frac{1-S_l}{S_l}\int_{-1}^{1}\phi_1^{(l)}\psi_2^{(n)}\,dt.$$

We can combine (4.20) and (4.21) to get

$$\begin{aligned}4\cos^4 S_n D_n(\exp(-S_n\tilde{D})-\exp(S_n\tilde{D}))-\sum_{l=-\infty}^{\infty}A_{ln}D_l\left(\exp(-S_l\tilde{D})+\frac{1+S_l}{1-S_l}\exp(S_l\tilde{D})\right)\\
=\frac{1}{4S_n^2}(1-\exp(-S_n\tilde{D}))+\sum_{l=-\infty}^{\infty}A_{ln}\frac{\exp(-S_l\tilde{D})}{16S_l^2\cos^4 S_l}\quad\text{for}\quad n=\pm1,\pm2,\ldots.\end{aligned}\tag{4.22}$$

We solved (4.22) by truncation and checked the convergence of the solution of the truncated equations numerically. In all cases considered by us, convergence is very rapid (see table 2). In deep trenches the factor $\exp(S_l\tilde{D})$ in the first term of (4.22) becomes large; it is convenient for computations in the deep trench to work with coefficients $\hat{D}_l=\exp(S_l\tilde{D})D_l$ which satisfy the relation

$$\begin{aligned}4\cos^4 S_n(\exp(-2S_n\tilde{D})-1)\hat{D}_n-\sum_{l=-\infty}^{\infty}A_{ln}\hat{D}_l\left(\exp(-2S_l\tilde{D})+\frac{1+S_l}{1-S_l}\right)\\
=\frac{1}{4S_n^2}(1-\exp(-S_n\tilde{D}))+\sum_{l=-\infty}^{\infty}A_{ln}\frac{\exp(-S_l\tilde{D})}{16S_l^2\cos^4 S_l}.\end{aligned}\tag{4.23}$$

The $\hat{D}_l$ are computed from (4.23); then the C_l are computed from (4.20).

We computed coefficients and plotted level lines of the edge stream function $\psi(t,y)$ satisfying (4.3) and the real stream function

$$\frac{\mu\psi}{\rho g\alpha d^3}=\psi(t,y)-\frac{(t^2-1)^2}{384},\tag{4.24}$$

for trenches with depth/width ratios of 5 (figure 2), 1 (figure 3) and $\frac{1}{2}$ (figure 4). The effect of the edge eddies at the top and bottom of the trench extends into the interior for a distance of about three times the width of the trench. The flow in the centre of the trench with depth/width ratio of 5 is nearly the same as the flow far below the surface in an infinitely deep trench.

t	$1000\dfrac{(t^2-1)^2}{384}$	$\Psi(t,0;\,1)\times 1000$	$\Psi(t,0;\,5)\times 1000$
0	2·6042	2·6194	2·6041
0·2	2·4000	2·4121	2·4000
0·4	1·8375	1·8401	1·8375
0·6	1·0667	1·0580	1·0666
0·8	0·3375	0·3280	0·3376
1·0	0	0	0
t	$1000\dfrac{(t^2-1)^2}{384}$	$\Psi(t,-\tilde{D};\,1)\times 1000$	$\Psi(t,-\tilde{D};\,5)\times 1000$
0	2·6042	2·5131	2·6042
0·2	2·4000	2·3446	2·4000
0·4	1·8375	1·8566	1·8375
0·6	1·0667	1·1298	1·0666
0·8	0·3375	0·3757	0·3375
1·0	0	0	0

TABLE 2. Convergence of the Papkovich–Fadle series in the trench of finite depth

$$\begin{bmatrix}\Psi(t,0;N)\\ \Psi(t,-\tilde{D};N)\end{bmatrix} = \sum_{\substack{-N\\ n\neq 0}}^{N}\frac{\phi_1^{(n)}}{S_n^2}\begin{bmatrix}C_n+\hat{D}_n\exp(-S_n\tilde{D})\\ C_n\exp(-S_n\tilde{D})+\hat{D}_n\end{bmatrix} \sim \frac{(t^2-1)^2}{384}\begin{bmatrix}1\\1\end{bmatrix}$$

To compute the streamlines of the flow in the deformed domain, it is necessary to compute the shape of the free surface. At this stage, it is only possible to deal with quantities defined in the reference domain.

5. The shape of the free surface

We turn next to the computation of the first coefficient $h^{\langle 1\rangle}(x)$ in the expansion (3.16) of the free surface in powers of ϵ. This coefficient is to be determined as the solution of the problem (3.14) subject to contact line or contact angle boundary conditions specified under (2.6). Equation (3.14) may be rewritten in terms of the stream function ψ as

$$\sigma\frac{d^2h^{\langle 1\rangle}}{dx_0^2}-\rho g h^{\langle 1\rangle} = -\Phi^{\langle 1\rangle}-2\mu\frac{\partial^2\psi}{\partial z_0\,\partial x_0}. \tag{5.1}$$

Under the same change of variables as leads to (4.3), and introducing the dimensionless height correction

$$H(t) = \frac{\sigma}{2\rho g\alpha d^3}h^{\langle 1\rangle}(x_0),$$

we may rewrite (5.1) as $\qquad H''-a^2H = -(\Psi_{,ty}+\mathscr{P}),\qquad$ (5.2)

$a^2 = \rho g d^2/4\sigma$. $\mathscr{P}$ is given, through integration of (4.4), as

$$\mathscr{P} = \frac{y}{32}-\sum_{-N}^{N}C_n\exp(S_n y)\cos S_n\sin S_n t + A$$
$$-\sum_{-N}^{N}\hat{D}_n\exp[-S_n(y+\tilde{D})]\,[(S_n\sin S_n+2\cos S_n)\sin S_n t+S_n t\cos S_n\cos S_n t]. \tag{5.3}$$

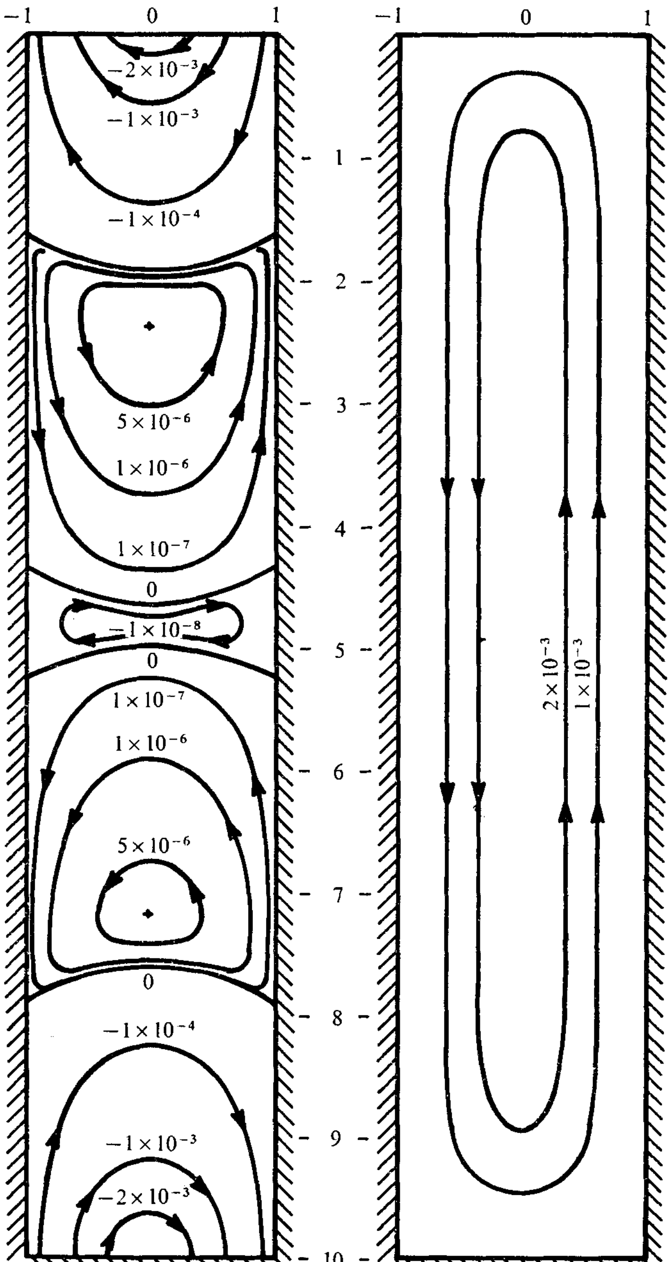

FIGURE 2. Level lines of the edge eddies (4.5) and streamlines (4.24) of the flow in the reference domain for the trench of finite depth with a depth/width ratio of five.

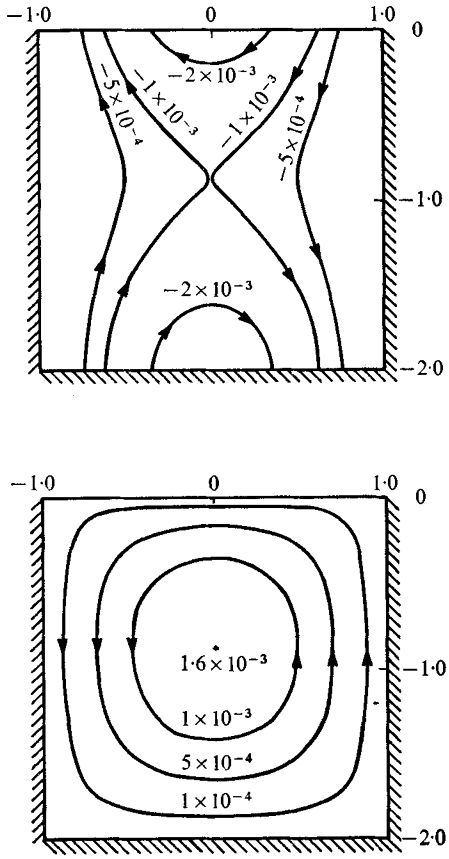

FIGURE 3. Level lines of the edge eddies (4.5) and streamlines (4.24) of the flow in the reference domain for the trench of finite depth with a depth/width ratio of 1.

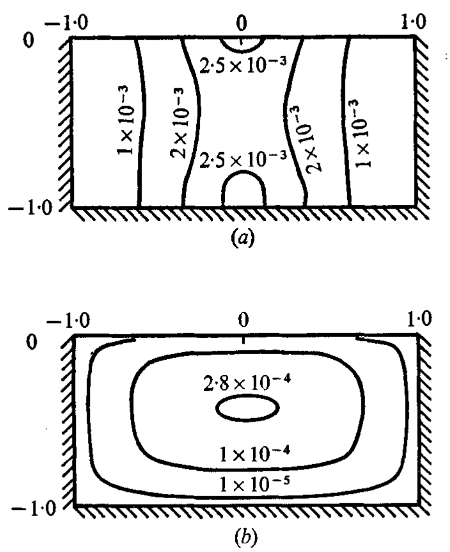

FIGURE 4. Level lines of the edge eddies (4.5) and streamlines (4.24) of the flow in the reference domain for the trench of finite depth with a depth/width ratio of $\frac{1}{2}$.

A is a constant to be determined. Equation (5.3) reduces to (4.17) when $\tilde{D} \to \infty$. Using (4.5) and (5.3), we rewrite (5.2) as

$$H'' - a^2 H + A = \sum_{-N}^{N} C_n[(S_n \sin S_n + 2\cos S_n)\sin S_n t + S_n t \cos S_n \cos S_n t] + \sum_{-N}^{N} \hat{D}_n \exp(-S_n \tilde{D}) \cos S_n \sin S_n t. \quad (5.4)$$

The general solution of (5.4) is

$$H = B_1 \exp(at) + B_2 \exp(-at) + \frac{A}{a^2} - \sum_{-N}^{N} C_n \left[\frac{(S_n \sin S_n + 2\cos S_n)}{a^2 + S_n^2} \sin S_n t + S_n \cos S_n \left(\frac{t \cos S_n t}{a^2+S_n^2} - \frac{2S_n \sin S_n t}{(a^2+S_n^2)^2}\right)\right] - \sum_{-N}^{N} \hat{D}_n \exp(-S_n \tilde{D}) \frac{\cos S_n}{a^2+S_n^2} \sin S_n t. \quad (5.5)$$

The last step in the evaluation of the height correction at first order is the application of the side conditions (2.6). In dimensionless variables, we may write the condition that $h(x)$ should have a zero mean value as

$$\int_{-1}^{1} H\, dt = 0. \quad (5.6)$$

When the trench is filled to the top with liquid, and the liquid adheres to sharp edge, we have

$$H(\pm 1) = 0. \quad (5.7)$$

From (5.5)–(5.7), we find that

$$A = 0, \quad B_1 = -B_2 = \tfrac{1}{2}b,$$

where

$$b = \frac{1}{\sinh a}\left\{\sum_{-N}^{N} C_n \left[\frac{S_n \sin S_n + 2\cos S_n}{a^2+S_n^2} \sin S_n + S_n \cos S_n \left(\frac{\cos S_n}{a^2+S_n^2} - \frac{2S_n \sin S_n}{(a^2+S_n^2)^2}\right)\right] + \sum_{-N}^{N} \hat{D}_n \exp(-S_n\tilde{D}) \frac{\cos S_n \sin S_n}{a^2+S_n^2}\right\},$$

and

$$H = -\sum_{-N}^{N} C_n \left\{\frac{S_n \sin S_n + 2\cos S_n}{a^2+S_n^2}\left(\sin S_n t - \frac{\sin S_n \sinh at}{\sinh a}\right) + \frac{S_n \cos S_n}{a^2+S_n^2}\left(t\cos S_n t - \frac{\cos S_n \sinh at}{\sinh a}\right) - \frac{2S_n^2 \cos S_n}{(a^2+S_n^2)^2}\left(\sin S_n t - \frac{\sin S_n \sinh at}{\sinh a}\right)\right\} - \sum_{-N}^{N} \hat{D}_n \exp(-S_n\tilde{D}) \frac{\cos S_n}{a^2+S_n^2}\left(\sin S_n t - \frac{\sin S_n \sinh at}{\sinh a}\right). \quad (5.8)$$

When the trench is partially filled with liquid and an angle of flat contact is specified,

$$H'(\pm 1) = 0, \quad (5.9)$$

we find, using (5.5), (5.6) and (5.9), that

$$A = 0, \quad B_1 = -B_2 = \tfrac{1}{2}b,$$

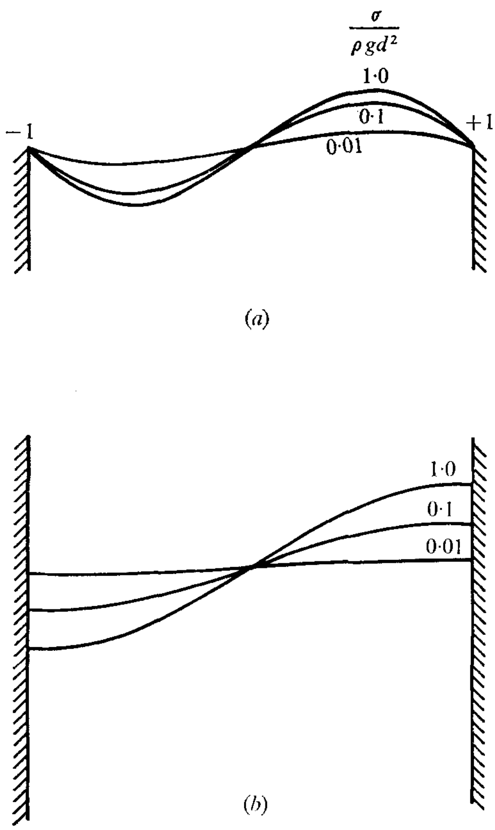

FIGURE 5. Graphs of the free-surface correction $H(t)$ for the infinitely deep trench. (*a*) The free surface on a liquid which adheres to the edge of the trench ((5.10) with $\tilde{D} = \infty$). (*b*) The free surface on a liquid when a flat slope of the free surface at the wall is prescribed.

where
$$b = \frac{1}{a\cosh a}\left\{\sum_{-N}^{N} C_n \left[\frac{S_n \sin S_n + 2\cos S_n}{a^2+S_n^2} S_n \cos S_n + S_n \cos S_n \left(\frac{\cos S_n - S_n \sin S_n}{a^2+S_n^2} - \frac{2S_n^2 \cos S_n}{(a^2+S_n^2)^2}\right)\right] + \sum_{-N}^{N} \hat{D}_n \exp(-S_n \tilde{D}) \frac{\cos S_n}{a^2+S_n^2} S_n \cos S_n\right\},$$

and
$$H = -\sum_{-N}^{N} C_n \left\{\frac{S_n \sin S_n + 2\cos S_n}{a^2+b^2}\left(\sin S_n t - \frac{S_n \cos S_n \sinh at}{a\cosh a}\right) + \frac{S_n \cos S_n}{a^2+S_n^2}\left(t\cos S_n t - \frac{(\cos S_n - S_n \sin S_n)\sinh at}{a\cosh a}\right) - \frac{2S_n^2 \cos S_n}{(a^2+S_n^2)^2}\left(\sin S_n t - \frac{S_n \cos S_n \sinh at}{a\cosh a}\right)\right\} - \sum_{-N}^{N} \hat{D}_n \exp(-S_n \tilde{D}) \frac{\cos S_n}{a^2+S_n^2}\left(\sin S_n t - \frac{S_n \cos S_n \sinh at}{a\cosh a}\right). \quad (5.10)$$

$4a^2$	H_{max} for $H(\pm 1) = 0$	H_{max} for $H'(\pm 1) = 0$
0·01	$3{\cdot}68 \times 10^{-4}$	$4{\cdot}06 \times 10^{-4}$
0·10	$1{\cdot}03 \times 10^{-3}$	$2{\cdot}55 \times 10^{-3}$
1·0	$1{\cdot}26 \times 10^{-3}$	$4{\cdot}75 \times 10^{-3}$
10·0	$1{\cdot}29 \times 10^{-3}$	$5{\cdot}19 \times 10^{-3}$
100·0	$1{\cdot}29 \times 10^{-3}$	$5{\cdot}24 \times 10^{-3}$

TABLE 3. The maximum value of the height rise coefficient $H(t;\, a^2)$ as a function of a^2. When the fluid sticks to the edge $H(\pm 1) = 0$, the maximum deflexion occurs for $0 < t < 1$. When an angle of flat contact is prescribed, $H'(\pm 1) = 0$, the maximum height rise is at $t = 1$

t	$H \times 10^3$ (given by (6.8))	$H \times 10^3$ (given by (6.10))
0	0·00	0·00
0·10	0·35	0·81
0·20	0·67	1·59
0·30	0·95	2·32
0·40	1·14	2·98
0·50	1·25	3·55
0·60	1·24	4·02
0·70	1·11	4·37
0·80	0·85	4·60
0·90	0·48	4·72
1·00	0·00	4·75

TABLE 4. The correction coefficients for the free surface on a liquid in a trench with a depth/width ratio of $\frac{1}{2}$. The first column is for prescribed contact line problem (5.8). The second column is for the prescribed contact angle problem with a flat contact

In figure 5 we have plotted the correction coefficient H for the free surface in an infinitely deep trench. (See table 3.) This coefficient is the dimensionless form of $h^{\langle 1\rangle}$; and the free surface is given by $h(x, \epsilon) = h^{\langle 1\rangle}\epsilon + O(\epsilon^2)$. Figure 5(*a*) gives the graph of H when the fluid grips the edge. Figure 5(*b*) gives the graph of H when the fluid surface is perpendicular to the side walls. Given $h^{\langle 1\rangle}$, we may give explicit form to the *shifting transformation*

$$z_0 \simeq h^{\langle 1\rangle}\epsilon - z. \tag{5.11}$$

Then, by inverting the mapping as in § 3, we may obtain, at lowest order, the form of the streamlines in the deformed domain.

In figure 6 we have carried out this inversion graphically. The streamlines of the flow in the reference domain are determined independently of the height rise, as in figure 6(*a*). To obtain the streamlines in the deformed domain, we must compute the shape of the free surface. Values of the function H, giving $h^{\langle 1\rangle}$, for a depth/width ratio of $\frac{1}{2}$ are given in table 4. Since the trench of figure 6 has a bottom, we obtain the stream function

$$\psi(x, a;\, \epsilon) = \psi^{[1]}(x_0, z_0)\,\epsilon + O(\epsilon^2),$$

where, by calculation,

$$\psi^{[1]}(x_0, z_0) = \psi^{\langle 1\rangle}(x_0, z_0),$$

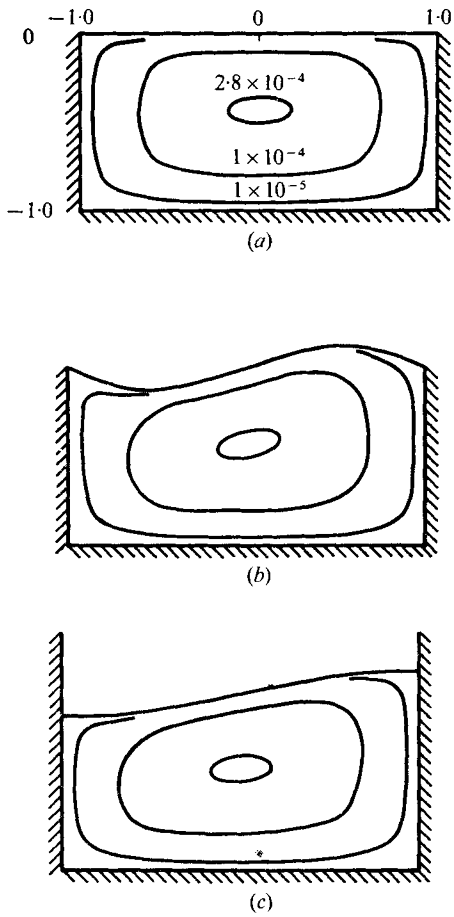

FIGURE 6. Streamlines in a trench with a depth/width ratio of $\frac{1}{2}$. (*a*) Streamlines in the reference domain. (*b*) Streamlines in a deformed domain when the liquid adheres to the edge of the trench. (*c*) Streamlines in a deformed domain when a flat slope at the wall is prescribed. The streamlines in (*b*) and (*c*) are the distortions of the level lines of (*a*) under the scaling transformation (5.12).

by inverting the *scaling transformation*

$$z_0 = \tilde{D}[h-z]/[\tilde{D}+h]. \tag{5.12}$$

Then
$$\psi(x,z;\,\epsilon) = \psi^{\langle 1\rangle}(x, \tilde{D}[h-z]/[\tilde{D}+h])\,\epsilon + O(\epsilon^2),$$

where
$$-\tilde{D} < z < h = h^{\langle 1\rangle}\epsilon + O(\epsilon^2).$$

The shape of the streamlines in the interior depends on ϵ through $h(x,\epsilon)$ alone. The inversion (5.12) of the stretching transformation will generate a different distortion of the level lines of $\psi^{\langle 1\rangle}(x_0, z_0)$ for each function $h(x;\,\epsilon) \simeq h^{\langle 1\rangle}\epsilon$. In figures 6 (*b*), (*c*), we have sketched two such distortions, corresponding to case 6 (*b*), in which the fluid grips the edge, and the case 6 (*c*), in which the fluid surface is perpendicular to the side walls. More flow patterns could be obtained from 6 (*a*), by generating different height corrections through varying the boundary conditions $h(x;\epsilon)$ at the side walls.

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Appendix. Domain perturbations for a simple example

Consider the following simple problem which we have invented to illustrate the concepts introduced in § 3:

$$\nabla^2 G(x,y;\epsilon) = 0 \quad \text{in} \quad \mathscr{V}_\epsilon = [x,y \mid -\infty < x < \infty,\ 0 \leqslant y \leqslant h(x;\epsilon)], \qquad \text{(A 1}a\text{)}$$

$$G(x,h;\epsilon) = \epsilon \sin x, \qquad \text{(A 1}b\text{)}$$

$$G(x,0;\epsilon) = G(x,y;\epsilon) - G(x+2\pi,y;\epsilon) = 0, \qquad \text{(A 1}c\text{)}$$

$$h(x;\epsilon) = D + \gamma \left.\frac{\partial G}{\partial y}\right|_{y=h} \equiv D + \gamma H(x,h(x;\epsilon);\epsilon). \qquad \text{(A 1}d\text{)}$$

γ is a constant. The linear *scaling transformation* $x = x_0,\ y = y_0 h(x;\epsilon)/D$ (see 3.2b) is used to map $\mathscr{V}_\epsilon$ onto

$$\mathscr{V}_0 = [x_0, y_0 \mid -\infty < x_0 < \infty,\ 0 \leqslant y_0 \leqslant D].$$

Series solutions of (A 1), analogous to (3.15) and (3.16), are

$$\begin{bmatrix} G(x,y;\epsilon) \\ h(x;\epsilon)-D \end{bmatrix} = \sum_{n=1} \frac{\epsilon^n}{n!} \begin{bmatrix} G^{[n]}(x_0,y_0) \\ h^{\langle n\rangle}(x_0) \end{bmatrix} = \sum \frac{\epsilon^n}{n!} \begin{bmatrix} G^{[n]}(x,yD/h) \\ h^{\langle n\rangle}(x) \end{bmatrix} = \sum_{n=1} \frac{\epsilon^n}{n!} \begin{bmatrix} G^{\langle n\rangle}(x,y) \\ h^{\langle n\rangle}(x) \end{bmatrix}, \qquad \text{(A 2)}$$

where
$$G^{[0]} = G^{\langle 0\rangle} = 0 \quad \text{and} \quad h^{\langle 0\rangle} = h^{[0]} = D. \qquad \text{(A 3)}$$

Substantial derivatives following the mapping are formed as in (3.4):

$$G^{[1]} = G^{\langle 1\rangle} + y^{[1]} G^{\langle 0\rangle}_{,y_0} = G^{\langle 1\rangle} + \frac{h^{\langle 1\rangle}}{D} y_0 G^{\langle 0\rangle}_{,y_0} = G^{\langle 1\rangle}(x_0,y_0), \qquad \text{(A 4)}$$

$$G^{[2]} = G^{\langle 2\rangle} + 2y^{[1]} G^{\langle 1\rangle}_{,y_0} + y^{[2]} G^{\langle 0\rangle}_{,y_0} + y^{[1]^2} G^{\langle 0\rangle}_{,y_0 y_0}$$
$$= G^{\langle 2\rangle}(x_0,y_0) + 2\frac{h^{\langle 1\rangle}}{D} y_0 G^{\langle 1\rangle}_{,y_0}(x_0,y_0). \qquad \text{(A 5)}$$

It is not hard to carry out computations to higher order; but the computation up to order two will suffice for our purpose.

The boundary-value problems for the function $G^{\langle n\rangle}(x_0,y_0)$, $n \geqslant 1$, are

$$\nabla^2 G^{\langle n\rangle} = 0 \quad \text{in} \quad \mathscr{V}_0,$$
$$G^{[n]}(x_0,D) = \delta_{n1} \sin x_0,$$
$$G^{\langle n\rangle}(x_0,0) = G^{\langle n\rangle}(x_0,y_0) - G^{\langle n\rangle}(x_0+2\pi,y_0) = 0,$$
$$h^{\langle n\rangle}(x_0) = \gamma H^{[n]}.$$

When $n = 1$,
$$\nabla^2 G^{\langle 1\rangle} = 0,$$
$$G^{[1]}(x_0,D) = G^{\langle 1\rangle}(x_0,D) = \sin x_0,$$
$$G^{\langle 1\rangle}(x_0,0) = G^{\langle 1\rangle}(x_0,y_0) - G^{\langle 1\rangle}(x_0+2\pi,y_0) = 0.$$

Hence, $$G^{\langle 1\rangle} = \sinh y_0 \sin x_0/\sinh D, \tag{A 6a}$$

and $$h^{\langle 1\rangle} = \gamma \coth D \sin x_0. \tag{A 6b}$$

When $n = 2$, $$\nabla^2 G^{\langle 2\rangle} = 0,$$

$$G^{[2]}(x_0, D) = G^{\langle 2\rangle} + 2h^{\langle 1\rangle} G^{\langle 1\rangle}_{,y_0} = G^{\langle 2\rangle}(x_0, D) + \gamma \coth^2 D \times (1 - \cos 2x_0) = 0,$$

$$G^{\langle 2\rangle}(x_0, 0) = G^{\langle 2\rangle}(x_0, y_0) - G^{\langle 2\rangle}(x_0 + 2\pi, y_0) = 0.$$

Hence, $$G^{\langle 2\rangle} = -\gamma \coth^2 D \left[\frac{y_0}{D} - \frac{\sinh 2y_0}{\sinh 2D}\cos 2x_0\right], \tag{A 7a}$$

$$G^{[2]} = G^{\langle 2\rangle} + \gamma \frac{y_0}{D} \coth D \frac{\cosh y_0}{\sinh D}(1 - \cos 2x_0), \tag{A 7b}$$

and $$h^{\langle 2\rangle}(x_0) = [\gamma H^{\langle 2\rangle} + 2\gamma h^{\langle 1\rangle} H^{\langle 1\rangle}_{,y_0}]|_{y_0 = D}. \tag{A 7c}$$

Now we are going to compare the two expressions for $G(x, y; \epsilon)$ given by (A 2) up to terms of order two. The series using the continuation of the functions $G^{\langle n\rangle}(x_0, y_0)$ into $\mathscr{V}_\epsilon$ is

$$\begin{aligned} G(x, y; \epsilon) &= \epsilon G^{\langle 1\rangle}(x, y) + \tfrac{1}{2}\epsilon^2 G^{\langle 2\rangle}(x, y) + O(\epsilon^3) \\ &= \epsilon \frac{\sinh Y}{\sinh D}\sin x - \tfrac{1}{2}\epsilon^2 \gamma \coth^2 D\left[\frac{y}{D} - \frac{\sinh 2y}{\sinh 2D}\cos 2x\right] + O(\epsilon^3). \end{aligned} \tag{A 8}$$

On the other hand, the series of substantial derivatives, using the mapping principle, is

$$\begin{aligned} G(x, y; \epsilon) &= \epsilon G^{[1]}(x, yD/h) + \tfrac{1}{2}\epsilon^2 G^{[2]}(x, yD/h) + O(\epsilon^3) \\ &= \epsilon \frac{\sinh yD/h}{\sinh D}\sin x - \tfrac{1}{2}\epsilon^2 \gamma \coth^2 D\left[\frac{y}{h} - \frac{\sinh 2yD/h}{\sinh 2D}\cos 2x\right] \\ &\quad + \tfrac{1}{2}\epsilon^2 \gamma \frac{y}{h}\coth D \frac{\cosh YD/h}{\sinh D}(1 - \cos 2x) + O(\epsilon^3). \end{aligned} \tag{A 9}$$

Equations (A 8) and (A 9) are representations of the same function. To recover (A 8) from (A 9), hold y fixed in (A 9), and expand $h(x; \epsilon)$ in powers of ϵ, using (A 2), (A $6b$) and (A $7c$).

On the boundary $y = h(x; \epsilon)$, (A 8) reduces to

$$G(x, h; \epsilon) = \epsilon \frac{\sinh h}{\sinh D}\sin x - \tfrac{1}{2}\epsilon^2 \gamma \coth^2 D\left[\frac{h}{D} - \frac{\sinh 2h}{\sinh 2D}\cos 2x\right] + O(\epsilon^3), \tag{A 10}$$

whereas, (A 9) reduces to

$$G(x, h; \epsilon) = \epsilon \sin x. \tag{A 11}$$

Superficially, it appears that the series (6.10) does not satisfy the boundary condition (A $1b$). However,

$$h = h(x; \epsilon) = \epsilon h^{\langle 1\rangle}(x) + \tfrac{1}{2}\epsilon^2 h^{\langle 2\rangle}(x) + O(\epsilon^3);$$

and, when (A 10) is re-expanded in powers of ϵ, it does take on the prescribed form.

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